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Analysis of a Cracked Thin Isotropic Plate Subjected to Bending Moment by Using FEAM

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The Finite element alternating method is applied to obtain the stress intensity factors of collinear multiple cracks in a thin isotropic plate subjected to bending moment. The necessary analytical solutions are obtained by using the complex stress function method given by Muskhelishvili and Savin. In order to verify the efficiency of the proposed method, several example problems are solved and compared with the published results.

Key Words : Finite Element Alternating Method, Stress Intensity Factor, Bending Moment, Complex Stress Function

1. Introduction

In order to assess the integrity of cracked structures, an accurate and effective stress analysis methodology is necessary. For two dimensional cracks under plane loading, it is verified that the finite element alternating method (FEAM) is a simple and efficient computational technique in obtaining the stress intensity factors (Park et al., 1992; Atluri, 1997) or in structural integrity assessment (Singh et al., 1994; Park et al., 1995). However usual structural plates sustain bending and twisting moments as well as plane loading, so the finite element alternating method need to be extended to consider such problems.

The problems of cracks in a thin plate subject to bending or twisting moments have been considered based on the classical theory (Sih et al., 1962; Isida, 1977; Merkulov, 1975; Lin'kov and Merkulov, 1975) or the Reissner theory (Murakami, 1987; Reissner, 1944, 1945). As well be known, the solutions obtained from the classical theory can not satisfy the exact physical boundary conditions on crack surfaces. On the other hand the Reissner plate theory can satisfy the exact boundary conditions but requires more complex solution procedure. Also the two theories give different stress intensity factors. But as indicated by Hui and Zehnder (1993), the mode I SIF values obtained from one theory can be converted into the SIF values of the other theory by using a simple relationship.

Chen et al. (1992, 1993) have considered the finite element alternating method in classical bending problems. They used the Fourier transformation technique in obtaining the necessary analytical solutions. In this paper, we are using other forms of analytical solutions obtained from the complex stress function method given in Muskhelishvili (1953) and Savin (1961) based on the classical theory. With this method, the Green functions are obtained for the collinear multiple cracks. The collinear multiple cracks subjected to arbitrary bending moment distribution on the crack surfaces can be analyzed by using the Green functions.

By inserting the obtained analytical solutions into the usual finite element alternating algorithm, the finite element alternating method is developed for bending problems. With the proposed method, the SIF values of collinear multiple cracks in a thin plate can be obtained very effectively.

In order to verify the efficiency of the method, several example problems are solved and compared with the published results.

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2. Formulation

In the classical plate theory, the governing equation for a thin isotropic plate under bending or twisting loading is reduced to a biharmonic equation. So the general solution can be written in terms of the complex stress functions by applying the method of Muskhelishvili(1953; Savin, 1961):

$$w(x, y) = Re[\bar{z}\varphi(z) + \chi(z)], \tag{1}$$

here w is the lateral deflection and $\chi'(z) = \psi(z)$.

The bending and twisting moments per unit length M_x, M_y, M_{xy} and the shear forces per unit length Q_x, Q_y are related with the stress functions as follows (Savin, 1961):

$$\begin{aligned} M_x + M_y &= -2D(1+\nu)[\Phi(z) + \overline{\Phi(z)}], \\ M_y - M_x + 2iM_{xy} &= 2D(1-\nu)[\bar{z}\Phi'(z) + \Psi(z)] \\ Q_x - iQ_y &= -4D\Phi'(z) \end{aligned} \tag{2}$$

where $\Phi(z) = \varphi'(z)$, $\Psi(z) = \psi'(z)$ and $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate. And E is the elastic modulus, h is thickness of a plate and ν is Poisson's ratio. Also the x and y direction displacements, u and v are related with the stress functions as follows:

$$u + iv = -\delta[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}] \tag{3}$$

here δ is the z coordinate.

Since only two boundary conditions can be imposed in the classical plate theory, the boundary conditions can be expressed as follows for the first fundamental problems:

$$\begin{aligned} M_n &= m(s), \\ Q_n + \frac{\partial M_{nt}}{\partial s} &= n(s). \end{aligned} \tag{4}$$

Here the subscript n denotes the normal direction and t denotes the tangental direction, and $m(s)$ and $n(s)$ are the given boundary values at s where s is the coordinate along the boundary. The second condition of Eq. (4) can be expressed as an integrated form as:

$$P + M_{nt} = f(s) + C^* \tag{5}$$

where $P'(s) = Q_n(s)$, $f'(s) = n(s)$ and C^* is a real integration constant. P can be related with the stress functions as $P = 2iD[\Phi(z) - \overline{\Phi(z)}]$.

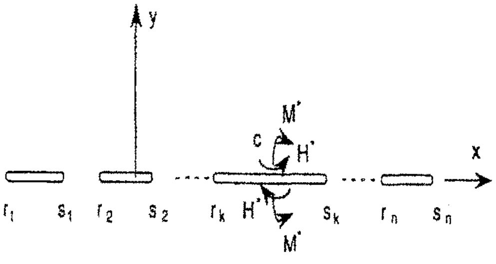

Fig. 1 Collinear multiple cracks subjected to bending and twisting moments at $x = c$.

Consider the problem of multiple cracks lying on the x axis as shown in Fig. 1. Each crack is of an arbitrary length. Arbitrarily distributed moments can be applied on the crack surfaces. And it is assumed that all moments are bounded at infinity. This problem can be formulated as the Hilbert problem by using the procedure given in Muskhelishvili(1953). A similar formulation can be found in Merkulov (1975) and Lin'kov and Merkulov (1975). The general solutions of the Hilbert problem can be obtained by using the results given in Muskhelishvili (1953) such as:

$$\begin{aligned} \Phi(z) &= \Phi_o(z) + \frac{P_n(z)}{X(z)} + \alpha \\ \Omega_b(z) &= \Omega_{bo}(z) + \frac{P_n(z)}{X(z)} - \alpha \end{aligned} \tag{6}$$

where

$$\begin{aligned} \Phi_o(z) &= \frac{1}{2\pi i X(z)} \int_L \frac{X(t)p(t)\,dt}{t-z} + \frac{1}{2\pi i} \int_L \frac{q(t)\,dt}{t-z} \\ \Omega_{bo}(z) &= \frac{1}{2\pi i X(z)} \int_L \frac{X(t)p(t)\,dt}{t-z} - \frac{1}{2\pi i} \int_L \frac{q(t)\,dt}{t-z} \end{aligned} \tag{7}$$

and

$$\Omega_b(z) = -\frac{(1-\nu)}{(3+\nu)}[\bar{\Phi}(z) + z\bar{\Phi}'(z) + \bar{\Psi}(z)] \tag{8}$$

$$p(t) = -\frac{1}{2D(3+\nu)}\{[M_y^+ + M_y^-] - i[(M_{xy}^+ - P^+) + (M_{xy}^- - P^-)]\} \tag{9}$$

$$q(t) = -\frac{1}{2D(3+\nu)}\{[M_y^+ - M_y^-] - i[(M_{xy}^+ - P^+) - (M_{xy}^- - P^-)]\} \tag{10}$$

$$P_n = C_0 z^n + C_1 z^{n-1} + \cdots + C_n, \tag{11}$$

$$X(z) = \prod_{j=1}^{n} \sqrt{z - r_j}\,\sqrt{z - s_j} \tag{12}$$

In Eqs. (6) and (11), α and C_0 are complex constants and can be determined by examining the stress and the behavior of the function $\Phi(z)$ at infinity. Other coefficients $C_1, C_2, \cdots, C_n$ can be obtained from the condition of the single valuedness of displacements u and v. Let Γ_j be a contour that surrounds the j-th crack. For the single valuedness of displacements, the lefthand side of Eq. (3) have to revert to its original value when z describes the contour Γ_j.

Once the stress functions are known, the stress intensity factors and stress fields can be calculated. K can be obtained from the following equations (Sih, 1962). For a righthand side crack tip;

$$K=-\frac{12\sqrt{2}D(3+\nu)}{h^2}\lim_{z\to s_j}\sqrt{z-s_j}\,\Phi(z) \quad (13)$$

where $K=K_1-iK_2$ and s_j is the x coordinate of the crack tip. For a lefthand side crack tip;

$$K=\frac{12\sqrt{2}D(3+\nu)}{h^2}\lim_{z\to r_j}\sqrt{r_j-z}\,\Phi(z) \quad (14)$$

where r_j is the x coordinate of the crack tip.

2.1 Multiple collinear cracks

Consider the problem where collinear multiple cracks exist in an infinite isotropic plate along the x axis. Each crack is of an arbitrary length. On the upper and lower crack surfaces of the kth crack, bending moments of magnitude M^* and twisting moments of magnitude H^* are applied at $x=c$ as shown in Fig. 1. And let the integration constant in Eq. (5) be zero. Then from Eq. (9) and Eq. (10), we can obtain:

$$p(t)=\frac{(M^*-iH^*)}{D(3+\nu)}\delta(t-c),$$
$$q(t)=0. \quad (15)$$

Here $\delta(t-c)$ is the Dirac delta function. Substituting Eq. (15) into Eq. (6), the stress functions become:

$$\Phi(z)=\Omega_b(z)=\frac{1}{2\pi i}\frac{(M^*-iH^*)}{D(3+\nu)}\frac{1}{X(z)}\times\left\{\frac{X(c)}{c-z}+i[c_1z^{n-1}+c_2z^{n-2}+\cdots+c_n]\right\}. \quad (16)$$

In order to obtain the coefficients in Eq. (16), we consider the condition of single valuedness of displacements u and v, which can be reduced to:

$$c_1\int_{r_j}^{s_j}\frac{t^{n-1}dt}{X(t)}+c_2\int_{r_j}^{s_j}\frac{t^{n-2}dt}{X(t)}+\cdots+c_n\int_{r_j}^{s_j}\frac{dt}{X(t)}=i\int_{r_j}^{s_j}\frac{X(c)}{X(t)}\frac{dt}{c-t}$$
$$j=1,2,\cdots,n \quad (17)$$

For $j=k$, the righthand side of Eq. (17) contains $(1/t)$ singularity in the integrand. In order to remove the difficulty in numerical integration, the following relation is used as in Park et al. (1992):

$$\int_{r_1}^{s_1}\frac{X(c)}{X(t)}\frac{dt}{c-t}+\cdots+\int_{r_j}^{s_j}\frac{X(c)}{X(t)}\frac{dt}{c-t}+\cdots+\int_{r_n}^{s_n}\frac{X(c)}{X(t)}\frac{dt}{c-t}=0. \quad (18)$$

After integrating each term in Eq. (17), we can obtain n linear algebraic equations, from which the coefficients $c_1, c_2, \cdots, c_n$ can be calculated.

From Eqs. (13) and (14), the SIF at the tips of multiple cracks can be obtained. For the crack tip at $x=r_j$:

$$K_1-iK_2=\frac{6\sqrt{2}(M^*-iH^*)}{\pi h^2}\frac{1}{X_2}\left[\frac{X_1}{\sqrt{c-r_j}}+iQ_n(r_j)\right] \quad (19)$$

where

$$X_1=\prod_{m=1,m\neq j}^{n}\sqrt{c-r_m}\prod_{m=1}^{n}\sqrt{c-s_m}$$
$$X_2=\prod_{m=1,m\neq j}^{n}\sqrt{r_j-r_m}\prod_{m=1}^{n}\sqrt{r_j-s_m}, \quad (20)$$
$$Q_n(z)=\sum_{m=1}^{n}c_mz^{n-m}.$$

And for the crack tip at $x=s_j$:

$$K_1-iK_2=\frac{6\sqrt{2}(M^*-iH^*)}{\pi h^2}\frac{1}{X_4}\left[\frac{X_3}{\sqrt{s_j-c}}-Q_n(s_j)\right] \quad (21)$$

where

$$X_3=\prod_{m=1}^{n}\sqrt{c-r_m}\prod_{m=1,m\neq j}^{n}\sqrt{c-s_m},$$
$$X_4=\prod_{m=1}^{n}\sqrt{s_j-r_m}\prod_{m=1,m\neq j}^{n}\sqrt{s_j-s_m}, \quad (22)$$

By using the solutions of this problem as Green functions, we can obtain the stress fields and SIF's for collinear multiple cracks, each of arbitrary length and each being subjected to arbitrary crack surface tractions.

The solutions given in Eq. (16) is obtained based on the assumption that the integration constant C^* in Eq. (5) equals to zero. If we remove this assumption, C^* must be determined from the condition of single valuedness of displacement w. But when $H^*=0$, we can verify that $C^*=0$. Since only Mode I loading, where $H^*=0$, is considered in this paper, solutions of Eq. (16) are enough to our purpose.

2.2 A single crack

When there is one crack of length $2a$ instead of multiple cracks, the solutions can be reduced from the results of multiple collinear cracks. The resulting stress functions become:

$$\Phi(z)=\Omega_b(z)=-\frac{1}{2\pi i}\frac{(M^*-iH^*)}{D(3+\nu)}\times\frac{\sqrt{c^2-a^2}}{\sqrt{z^2-a^2}}\frac{1}{z-c},$$
$$\varphi(z)=\omega_b(z)=\frac{1}{2\pi i}\frac{(M^*-iH^*)}{D(3+\nu)}\times\left[\log\frac{2\sqrt{c^2-a^2}\sqrt{z^2-a^2}+2cz-2a^2}{z-c}\right]. \quad (23)$$

Here we use the assumption that the integration constant C^* in Eq. (5) equals to zero. From these stress functions the SIF is expressed as:

$$K_1-iK_2=\frac{1}{\pi\sqrt{a}}\sqrt{\frac{a+c}{a-c}}\left[\frac{6(M^*-iH^*)}{h^2}\right]. \quad (24)$$

Next consider the case when the integration constant in Eq. (5) is not equal to zero. Then $p(t)$ and $q(t)$ can be expressed as:

$$p(t)=\frac{(M^*-iH^*)}{D(3+\nu)}\delta(t-c)+Ci,$$
$$q(t)=0. \quad (25)$$

Also the stress functions become:

$$\Phi(z)=\Omega_b(z)=-\frac{1}{2\pi i}\frac{(M^*-iH^*)}{D(3+\nu)}\times\frac{\sqrt{c^2-a^2}}{\sqrt{z^2-a^2}}\frac{1}{z-c}+\frac{Ci}{2}\left(1-\frac{z}{\sqrt{z^2-a^2}}\right). \quad (26)$$

Applying the condition of single valuedness of displacement w (Merkulov, 1975), we can obtain $C=-2H^*i\sqrt{c^2-a^2}/[\pi D(3+\nu)a^2]$. And the resulting stress functions become:

$$\Phi(z)=\Omega_b(z)=-\frac{1}{2\pi i}\frac{(M^*-iH^*)}{D(3+\nu)}\times\frac{\sqrt{c^2-a^2}}{\sqrt{z^2-a^2}}\frac{1}{z-c}+\frac{H^*\sqrt{c^2-a^2}}{\pi D(3+\nu)a^2}\left(1-\frac{z}{\sqrt{z^2-a^2}}\right). \quad (27)$$

From Eq. (13) the resulting SIF's become:

$$K_1-iK_2=\frac{1}{\pi\sqrt{a}}\sqrt{\frac{a+c}{a-c}}\frac{6}{h^2}\left[M^*-\frac{(2c-a)}{a}iH^*\right]. \quad (28)$$

The same result can be found in Merkulov (1975) and van Vroonhoven (1994).

Comparing the two results of Eq. (24) and Eq. (28), the same SIF values are obtained for the bending moment M^*, but different values for the twisting moment H^*. If we consider mixed mode problems it must be determined which solution must be used as Green functions. The solutions of Eq. (23) can not satisfy the condition of single valuedness of displacement w contrary to Eq. (27). But the stress field obtained from Eq. (27) is not uniquely determined by stress intensity factors but depends on the geometry also. In this paper, however, since we consider only Mode I loading, where $H^*=0$, all the solutions satisfy the single valuedness of displacement w.

3. Finite Element Alternating Method

A general and detailed description of the finite element alternating method can be found in Atluri (1986, 1997). The basic steps in the finite element alternating method for bending problems are the same as the usual procedure in plane problems. So several comments instead of detail descriptions are given here.

The necessary analytical solutions for a cracked infinite plate subjected to arbitrarily distributed bending moments on crack surfaces are obtained by using the solutions of Eq. (16) as the Green functions. Here the numerical integration is carried out by using the Gaussian type quadrature. When there is $1/\sqrt{r}$ type singularity in the integrands, the suitable Gaussian type integration formula given in Abramowith et al. (1972) is used.

Since we consider only symmetric problems only M_y moments are considered as residuals on crack surfaces. At outer boundaries, the residual

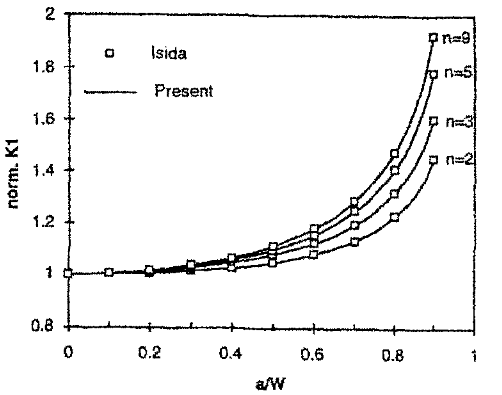

Fig. 2 Normalized SIF values of collinear multiple cracks. Here n is the number of cracks.

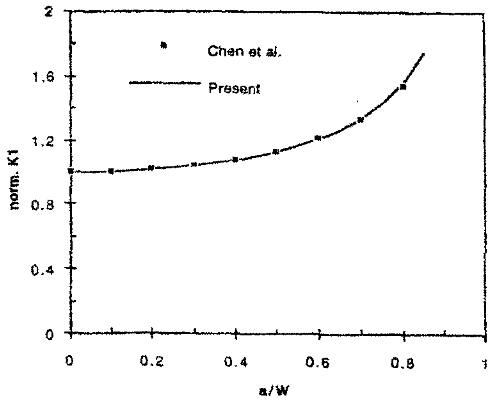

Fig. 3 Normalized SIF values of a center crack subject to remote bending moment.

forces from shear stress τ_{nz} and moments M_n, M_{nt} are considered.

The used finite element formulation and the finite element program are those given in Hinton and Owen (1977). Eight nodes plate elements are used and each node has three degree of freedoms of displacement w, x rotation and y rotation.

4. Sample Problems

In order to verify the efficiency of the proposed method, several sample problems are solved. First, in order to check the accuracy of the analytical solutions, we consider the problem of an infinite plate with equal collinear multiple cracks subjected to remote constant bending moment M_o. Here we assume that the cracks are periodically distributed, the distance between the centers of the adjacent cracks is $2d$, and the length of each crack is $2a$. The results are given in Fig. 2 when the number of cracks are 2, 3, 5, and 9. Here the SIF values are normalized with $6M_o\sqrt{a}/h^2$. Comparing with the results of Isida (1977), we can notice that the two values coincide very well.

Next consider the problem of a finite rectangular plate with a center crack of length $2a$. Constant bending moment M_o is applied on the horizontal edge. The plate has the height of $2H$, the width of $2W$ and the thickness of h. It is assumed that H is equal to W. The normalized SIF values are given as a function of a/W and compared with the results of Chen et al. (1992) in

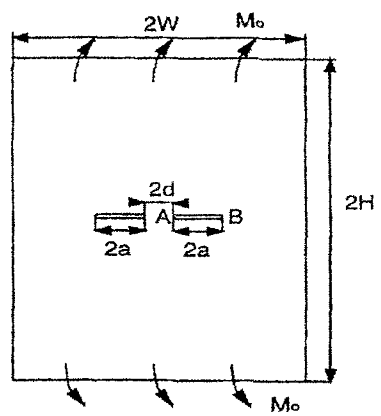

Fig. 4 Two symmetric cracks in a thin plate.

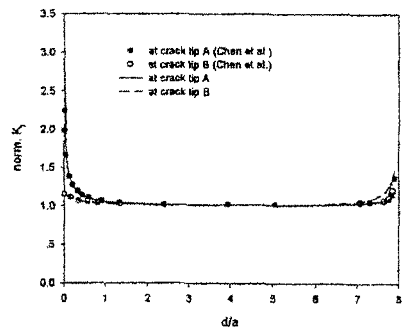

Fig. 5 Normalized SIF values of two cracks in a thin plate shown in Fig. 4.

Fig. 3. It can be shown that the two values coincide well.

Consider another problem of a finite rectangular plate with symmetrically located two cracks as shown in Fig. 4. In order to compare with the results of Chen et al. (1992) the same dimensions are used such as $H=3W$, $W=10h$ and $a=h$. The normalized SIF values are given as functions of d/a in Fig. 5. The results show good agreement with the results in Chen et al. (1992).

5. Results

The finite element alternating method is extended to consider collinear multiple cracks in a finite thin plate subjected to bending moments. It is shown that the proposed method can be used as an effective method in obtaining the stress intensity factors.

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